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INSTITUTE FOR DEFENSE ANALYSES ARLINGTON VA PROGRAM --ETC F/G 12/2  
A CLASS OF LANCHESTER ATTRITION PROCESSES. (U)

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**PAPER P-1230**

**CLASS OF LANCHESTER ATTRITION PROC**

**Alex R. Kerr**

**December 1970**

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The work reported in this publication was conducted under IDA's Information Resource Management (IRM) program and funded by the U.S. Agency for International Development (USAID). The views expressed in this document are those of the author(s) and do not necessarily reflect the official position of USAID.

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# **A CLASS OF LANCHESTER ATTRITION PROCESSES**

**Alan F. Karr**

**December 1976**

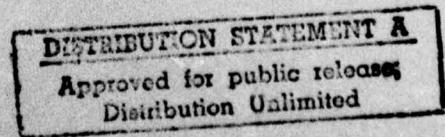


**INSTITUTE FOR DEFENSE ANALYSES**

**PROGRAM ANALYSIS DIVISION**

**400 Army-Navy Drive, Arlington, Virginia 22202**

**IDA Independent Research Program**



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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER <b>P-1230</b>	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) <b>A CLASS OF LANCHESTER ATTRITION PROCESSES</b>		5. REPORTS ON SAME SUBJECT COVERED <b>Final <i>rept. 2</i></b>
6. AUTHOR(s) <b>Alan F. Karr</b>	7. PERFORMING ORGANIZATION NAME AND ADDRESS <b>INSTITUTE FOR DEFENSE ANALYSES Program Analysis Division 400 Army-Navy Drive, Arlington, VA 22202</b>	8. CONTRACT OR GRANT NUMBER(s) <b>IDA Independent Research</b>
9. CONTROLLING OFFICE NAME AND ADDRESS	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS	
11. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	12. REPORT DATE <b>December 1976</b>	13. NUMBER OF PAGES <b>021 43 p.</b>
14. SECURITY CLASS (if in Report) <b>UNCLASSIFIED</b>	15. DECLASSIFICATION/DOWNGRADING SCHEDULE <b>N/A</b>	
16. DISTRIBUTION STATEMENT (of this Report)  This document is unclassified and suitable for public release.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)  <b>(18) IDA/HQ (19) 76-18674</b>		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)  Combat Model, Attrition Model, Lanchester Equations, Stochastic Attrition Equation, Markov Attrition Process		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number)  This paper derives from physical assumptions a class of stochastic attrition processes, some of which are analogous to the deterministic models of F. Lanchester. Shooting weapons are classified as having independent or proportional engagement initiation and as being single-kill or multiple kill. Several special cases of the generalized process are examined in detail and related to previous attrition models. Computational problems, fire allocation, and some related over		

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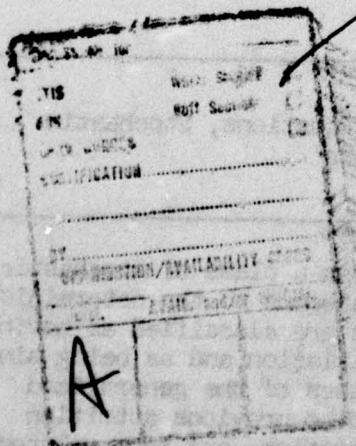
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## 1. INTRODUCTION AND PURPOSE

The principal goal of the research effort reported here is to derive certain stochastic attrition models from carefully stated probabilistic assumptions on behavior of individual combatants and on interactions among combatants. Although several processes derived appear to be stochastic analogues of the so-called "Lanchester equations of combat," in this paper we do not deal with the square law/linear law dichotomy that has so long engendered both debate and confusion. Our explicit identification of the assumptions underlying a number of stochastic models of combat attrition permits determination of the applicability of a given model in terms of those assumptions. If a combat situation is felt to be well described by the assumptions, then the model is appropriate. Hence, there is no need to maintain the poorly understood and misleading terminology and interpretation of "square law processes" and "linear law processes." However, the processes derived here we call LANCHESTER processes in honor of Lanchester's fundamental contributions to the subject.

The paper is organized in the following manner. Section 2 describes in detail the assumptions underlying a very general stochastic attrition process, and then characterizes that process as a Markov process. In Section 3 we obtain a number of specific (and, in some cases previously known) stochastic attrition models as special cases of the general model of Section 2. Some of these models are analogous to the deterministic Lanchester models; such analogies are indicated in order that the reader may relate our work to previous work

on mathematical modeling of combat processes. Finally, Section 4 deals with some possible extensions of the processes described here, computational aspects, and some additional problems of interest.

## 2. THE MATHEMATICAL MODEL

In this section we derive from a set of carefully stated and detailed probabilistic assumptions a very general stochastic model of combat between heterogeneous forces. The model is based on a classification of shooting weapons into four classes; the four classes are the result of two independent dichotomies of shooting weapons. First, shooting weapons are classified in terms of the qualitative nature of the rate at which opposition weapons are engaged. Here "rate" is in the sense of the infinitesimal generator of a continuous time Markov process, as described in more detail below. A particular type of shooting weapon will be said to have independent engagement initiation if the mean total rate at which it engages (fires upon) opposition weapons is independent of both the numerical size and the precise structure of the opposition force. In other words, the mean engagement rate of a weapon with independent engagement initiation depends only on the type of weapon. The possibility that the rate at which opposition weapons of a particular type are engaged may depend on the size and structure of the opposition force is not excluded; the definition is based only on the total rate of engagement. On the other hand, a shooting weapon will be said to have proportional engagement initiation if the mean total rate at which it engages opposition weapons in a

target force  $y = (y_1, \dots, y_n)$  is of the form  $\sum_{j=1}^n d_j y_j$  where  $y_j$  is the number of opposition weapons of type  $j$  present, and the  $d_j$  are nonnegative constants. In particular, the rate at which opposition weapons of a given type are engaged is directly proportional to the number of weapons of that type currently surviving.

We assume that every type of weapon falls into one of the two classes defined above. For some types of weapons there is little difficulty with this assumption. For example, an artillery piece seems rather clearly to possess independent engagement initiation, while a hand-held antitank weapon possesses proportional engagement initiation. But in other cases (tanks, for example) the problem is more difficult. In particular, the category to which a tank belongs might be the result of tactical decisions during the course of a battle, might change over time, and might even depend on the structure of the opposing force. Our model is incapable of representing such phenomena. In general, physical situations compatible with independent engagement initiation seem to be of two types. Either a shooting weapon simply fires (at a probabilistically constant rate) at an area in which opposition weapons are known or thought to be located, or a shooting weapon is mobile and moves forward (or is pushed backward) in such a manner as to maintain a constant (overall) rate of engagement with opposition forces. In this case every opposition weapon encountered is engaged. Weapons with proportional engagement initiation are those in which detection of an opposition weapon must precede its being engaged and in which the time required to detect a given opposition weapon is exponentially distributed. For such weapons every detected opposition weapon is engaged. There exists, of course, the possibility that some weapon fits well into neither class; the model is unable to accommodate this.

The second classification of shooting weapons is based on the maximum number of opposition weapons that can be destroyed in a single engagement. If that number is one, the shooting weapon is said to be single-kill; if the maximum is more than one, the shooting weapon is termed a multiple-kill weapon. As before, this classification is independent of the type of target, so the assumption is restrictive to the extent that a particular

type of weapon may in reality be single-kill against some types of targets but multiple-kill against others. In general, the property of being a single-kill weapon appears to be compatible with both independent and proportional engagement initiation, while the property of being a multiple-kill weapon is somewhat more compatible with the former. However, multiple-kill and proportional engagement initiation are not incompatible; consider a weapon that makes no shots until an opposition weapon is detected, but then makes a shot that is sufficiently powerful to destroy many opposition weapons.

According to our classification scheme there are four qualitative classes of weapons:

- IS - Weapons with independent engagement initiation and single kills per engagement;
- PS - Weapons with proportional engagement initiation and single kills per engagement;
- IM - Weapons with independent engagement initiation and multiple kills per engagement;
- PM - Weapons with proportional engagement initiation and multiple kills per engagement.

We next present the assumptions and notations for our generalized attrition process. Further discussion of the basic classification appears in the comments on the assumptions. The combat involves two heterogeneous forces (Red and Blue); weapon types are distinguished only into the four qualitative classes described above and, within each class, by variations of numerical parameters.

#### Assumptions

1. The Blue force consists of  $M_1$  weapon types of the class IS,  $M_2$  weapon types of class PS,  $M_3$  weapon types of the class IM, and  $M_4$  weapon types of the class PM. Hence, the Blue side has altogether  $M = M_1 + M_2 + M_3 + M_4$  types of weapons. The analogous numbers for the Red side are  $N_1, N_2, N_3, N_4$ , and  $N = N_1 + N_2 + N_3 + N_4$ .

The notation used below is of the following form: the letter "i" is used as a generic index for Blue weapon types. Types 1, ...,  $M_1$  are the IS weapons;  $M_1 + 1, \dots, M_1 + M_2$  are PS weapons; weapon types  $M_1 + M_2 + 1, \dots, M_1 + M_2 + M_3$  are of class IM; and the remaining types belong to the class PM. Certain parameters below are defined only for Blue weapons belonging to one class, e.g., if that class is IM, the relevant index  $i$  ranges over 1, ...,  $M_3$ ; when all Blue weapon types are considered, the index  $i$  ranges over 1, ...,  $M$ . For Red, a similar situation holds for the index  $j$ .

Blue forces are denoted by vectors  $x \in \mathbb{N}^M$ , where  $\mathbb{N} = \{0, 1, 2, \dots\}$ ;  $x_i$  is the number of type- $i$  weapons ( $i = 1, \dots, M$ ) currently present. Red forces are, analogously, denoted by vectors  $y \in \mathbb{N}^N$ .

2. Times between engagements initiated by a surviving Blue IS type- $i$  weapon are independent and identically exponentially distributed with mean  $1/r_B(i)$ ,  $i = 1, \dots, M_1$ .
3. When a Blue IS weapon of type  $i$  initiates an engagement, it attacks exactly one Red weapon. If the current Red force is the vector  $y$  the probability that the weapon attacked is of type  $j$  is  $a_B(i, y; j)$ ; the particular type- $j$  weapon attacked is chosen uniformly. Both choices are independent of the past history of the process and of each other.
4. The conditional probability that a Blue IS type- $i$  weapon kills a Red type- $j$  weapon, given an attack on that weapon, is  $p_B(i, j)$  for  $i = 1, \dots, M_1$  and  $j = 1, \dots, N$ .
5. Assumptions 2, 3, and 4 hold for Red IS weapons with parameters  $r_B(j)$ ,  $j = 1, \dots, N_1$ ,  $a_R(j, x; i)$ ;  $j = 1, \dots, N_1$ ;  $i = 1, \dots, M$  and  $x \in \mathbb{N}^M$ , and  $p_R(j, i)$ ,  $j = 1, \dots, N_1$ ;  $i = 1, \dots, M$ .

6. The time required for a particular surviving Blue PS type- $i$  weapon to detect a particular surviving Red type- $j$  weapon is exponentially distributed with mean  $1/d_B(i,j)$  for  $i = 1, \dots, M_2$  and  $j = 1, \dots, N$ . A Blue PS weapon detects different Red weapons independently of one another and can engage an opposition weapon only after detecting it.
7. A Blue PS weapon engages every Red weapon it detects. The conditional probability that a Blue PS type- $i$  weapon kills a Red type- $j$  weapon, given detection and attack, is  $k_B(i,j)$  for  $i = 1, \dots, M_2; j = 1, \dots, N$ .
8. Red PS weapons satisfy Assumptions 6 and 7 with mean detection times  $1/d_R(j,i)$  and kill probabilities  $k_R(j,i)$ , defined for  $j = 1, \dots, N_2$  and  $i = 1, \dots, M$ .
9. Times between engagements initiated by a surviving Blue IM type- $i$  weapon ( $i=1, \dots, M_3$ ) are independent and identically exponentially distributed with mean  $1/r_B^*(i)$ .
10. Given that a Blue IM type- $i$  weapon initiates an engagement against a currently surviving Red force  $y$ , the probability that the surviving target force has composition  $z$  is  $\varphi_B(i,y;z)$ . The numbers of surviving Red weapons are otherwise independent of the past history of the attrition process. Symbolically,

$$\varphi_B(i,y;z) = P\{\text{surviving Red force is } z \mid \text{engagement initiated by Blue IM type-}i \text{ weapon against Red force with composition } y\}.$$

Here,  $y, z \in \mathbb{N}^N$ .

11. Red IM weapons obey Assumptions 9 and 10, with mean engagement times  $1/r_R^*(j)$ ,  $j = 1, \dots, N_3$ , and kill distributions  $\varphi_R(j,x;w)$  defined for  $j = 1, \dots, N_3; x, w \in \mathbb{N}^M$ .

12. The time required for a particular surviving Blue PM type- $i$  weapon ( $i=1, \dots, M_4$ ) to detect a particular surviving Red type- $j$  weapon ( $j=1, \dots, N$ ) is exponentially distributed with mean  $1/d_B^*(i,j)$ . A Blue PM weapon detects different Red weapons independently of one another.
13. At the instant of each detection, a Blue PM weapon initiates an engagement.
14. Each such engagement is independent of the previous history of the attrition process. If a Blue PM type- $i$  weapon engages a Red force of composition  $y$  after having detected a Red type- $j$  weapon ( $j=1, \dots, N$ ), the probability that the Red force surviving the engagement is  $z$  is denoted by  $\mu_B(i,j,y;z)$ .
15. Red weapons of the class PM satisfy Assumptions 12, 13, and 14 with parameters  $d_R^*(j,i)$  and  $\mu_R(j,i,x;w)$  defined for  $j = 1, \dots, N_4$ ;  $i = 1, \dots, M$ , and  $x, w \in \mathbb{N}^M$ .
16. The detection, engagement, and kill processes of all weapons are mutually independent.

To avoid unnecessary proliferation of notation, certain parameters are denoted by an asterisk for multiple-kill weapons and no asterisk for single-kill weapons.

#### Comments on the Assumptions

It follows from Assumptions 2, 6, 9, 12, and 16 that the time required for a given shooting weapon to initiate an engagement is, conditioned on survival of the weapon until that time, exponentially distributed. The parameter of that exponential distribution depends on the class and type of the shooting weapon, but is one of two forms according as the weapon has independent or proportional engagement initiation. Table 1 illustrates for single-shot weapons of type  $i$  on the Blue side against a Red force  $y$ . The same holds for multiple-shot weapons but with parameters having asterisks.

Table 1. EXPONENTIAL PARAMETERS FOR  
OVERALL ENGAGEMENT RATES

Form of Engagement	Parameter
Initiation	
Independent	$r_B(i)$
Proportional	$\sum_j d_B(i,j)y_j$

Recalling that the parameter of an exponential distribution is the inverse of the expectation and that the latter is in units of time, we may interpret the entries in Table 1 as expected overall rates of engagement of the Red force by the weapon in question. Expected rates of engagement of particular types of Red weapons are given in Table 2.

Table 2. ENGAGEMENT RATES FOR PARTICULAR  
OPPOSITION WEAPON TYPES

Form of Engagement	Rate of Engagement of Opposition Weapons of Type j
Initiation	
Independent	$r_B(i) a_B(i,y;j)$
Proportional	$d_B(i,j)y_j$

Derivation of the kill distributions  $\varphi_B$ ,  $\varphi_R$ ,  $\mu_B$  and  $\mu_R$  is a problem that we have not yet considered in depth. In application of our model to computerized simulations, this would be the problem most in need of attention. Here we have indicated abstractly functional dependences that seem plausible and, hence, those that we feel can safely be ignored. For example, the kill distributions do not depend on the structure of the force to which a shooting weapon belongs, though in principle they could.

Indeed, a reasonable way of describing such dependence might allow representation of the synergistic interactions of weapons on the same side. Similarly, the kill distributions for PM weapons indicate that the distribution of weapons killed can depend on the particular (type of) weapon first detected; differing ammunition or tactics used against different detected weapons can thus be modeled.

In Section 4 we discuss at some length certain plausible forms for the fire allocation distributions  $a_B$  and  $a_R$ . In particular, all distributions satisfying some rather weak restrictions are of a certain simple form.

All "engagements" occur instantaneously, with ensuing total loss of contact. This is admittedly an unrealistic feature of the model, although most other models do not seem to have successfully addressed the difficulty either. One way of including binary engagements with exponentially distributed lengths is discussed in [8].

### Results

Our notation and terminology concerning Markov processes are those of [2], [3] and [6]; the reader is referred to these for background material. We recall here the following facts. Let  $(X_t)$  be a continuous time Markov process with countable state space and no instantaneous states. The matrix  $A$  defined by

$$A(i,j) = \lim_{t \rightarrow 0} \frac{P\{X_t = j | X_0 = i\} - I(i,j)}{t},$$

where  $I$  denotes the identity matrix, is called the infinitesimal generator of  $(X_t)$ . If

$$P_t(i,j) = P\{X_t = j | X_0 = i\}$$

\*

denotes the transition function of  $(X_t)$ , then  $A$  satisfies the backward equation

$$P_t' = AP_t$$

and the forward equation

$$P_t' = P_t A. \quad (1)$$

Matrices are differentiated componentwise with respect to  $t$ .

From (1) it follows that

$$P_{t+h}(i,j) \sim h \sum_k P_t(i,k)A(k,j)$$

for small values of  $h$ , so that  $A(k,j)$  may be interpreted as the (infinitesimal) rate at which the process tends to move from state  $k$  to state  $j$ . The quantity  $\lambda(i) = -A(i,i)$  is finite, but possibly zero, for each  $i$ . When the process enters state  $i$  it remains there for an exponentially distributed random time that is independent of the entire past history of the process and has expectation  $\lambda(i)^{-1}$ . If  $\lambda(i) = 0$ , then, once entered, state  $i$  cannot be left and is called absorbing. If  $\lambda(i) > 0$ , then state  $i$  is called stable; at the end of the sojourn the next state to be entered is state  $j$  with probability  $Q(i,j) = A(i,j)/\lambda(i)$ . The sequence of states entered is thus a Markov chain, called the embedded Markov chain, with transition matrix  $Q$ . We refer the reader to the previously mentioned sources for further details.

We can now describe and characterize the stochastic attrition process engendered by Assumptions 1 through 16 above. Let  $E = \mathbb{N}^{M+N}$  consist of states denoted by  $\alpha = (x,y)$ ;  $\alpha$  is to be thought of as a possible pair of surviving forces at some instant--with  $x$  describing the Blue force and  $y$  the Red. The sample space for our attrition process is the family  $\Omega$  of functions from  $\mathbb{R}_+$  to  $E$  that are right-continuous and have limits from the left with respect to the discrete topology on  $E$ . The

coordinate (vector-valued) stochastic process  $\{(B_t, R_t)\}_{t \geq 0}$  (here  $B_t: \Omega \rightarrow \mathbb{N}^M$  and  $R_t: \Omega \rightarrow \mathbb{N}^N$  for each  $t$ ) has the usual interpretation:  $B_t$  is the surviving Blue force at time  $t$ ;  $R_t$ , the Red force at time  $t$ . We further define

$$\underline{E}_t = \sigma\{(B_u, R_u); 0 \leq u \leq t\},$$

which is the history of the attrition process until time  $t$ , and

$$\underline{E} = \sigma\{(B_u, R_u); u \geq 0\},$$

which is the entire history of the process. For each  $\alpha \in E$ , we denote by  $P^\alpha$  the probability law on  $(\Omega, \underline{E})$  of the attrition process governed by Assumptions 1 through 16, subject to the initial condition

$$P^\alpha\{(B_0, R_0) = \alpha\} = 1.$$

Here is our main result:

**THEOREM.** Subject to Assumptions 1 through 16 listed above, the stochastic process

$$(B, R) = (\Omega, \underline{E}, \underline{E}_t, (B_t, R_t), P^\alpha) \quad (2)$$

is a Markov process with state space  $E$ . The infinitesimal generator  $A$  of the process is of the following form. Consider two states  $\alpha = (x, y)$  and  $\alpha' = (x', y')$  such that  $y' = y$ ,  $x'_i = x_i - 1$  for some  $i$  and  $x'_k = x_k$  for all  $k \neq i$  (the new state  $\alpha'$  is reached from  $\alpha$  by the destruction of *exactly one* Blue type- $i$  weapon); then

$$\begin{aligned}
A(\alpha, \alpha') = & \sum_{j=1}^{N_1} r_R(j) a_R(j, x; i) p_R(j, i) y_j \\
& + x_1 \sum_{j=1}^{N_2} d_R(j, i) k_R(j, i) y_{N_1+j} \\
& + \sum_{j=1}^{N_3} r_R^*(j) \varphi_R(j, x; x') y_{N_1+N_2+j} \\
& + \sum_{k=1}^M x_k \sum_{j=1}^{N_4} d_R^*(j, k) \mu_R(j, k, x; x') y_{N_1+N_2+N_3+j} . \quad (3)
\end{aligned}$$

For any other state  $\alpha' = (x', y')$  with  $y' = y$ ,  $x'_i \leq x_i$  for all  $i$ , and  $\sum_i (x_i - x'_i) \geq 2$ , which corresponds to the simultaneous destruction of more than one Blue weapon and can be effected only by Red weapons of classes IM and PM, we have

$$\begin{aligned}
A(\alpha, \alpha') = & \sum_{j=1}^{N_3} r_R^*(j) \varphi_R(j, x; x') y_{N_1+N_2+j} \\
& + \sum_{k=1}^M x_k \sum_{j=1}^{N_4} d_R^*(j, k) \mu_R(j, k, x; x') y_{N_1+N_2+N_3+j} . \quad (4)
\end{aligned}$$

Similarly, for  $\alpha' = (x', y')$  such that  $x' = x$ ,  $y'_j = y_j - 1$  and  $y'_\ell = y_\ell$  for  $\ell \neq j$ ,

$$\begin{aligned}
A(\alpha, \alpha') = & \sum_{i=1}^{M_1} r_B(i) a_B(i, y; j) p_B(i, j) x_i \\
& + y_j \sum_{i=1}^{M_2} d_B(i, j) k_B(i, j) x_{M_1+i} \\
& + \sum_{i=1}^{M_3} r_B^*(i) \varphi_B(i, y; y') x_{M_1+M_2+i}
\end{aligned}$$

$$+ \sum_{\ell=1}^N y_{\ell} \sum_{i=1}^{M_4} d_B^*(i, \ell) \mu_B(i, \ell, y; y') x_{M_1+M_2+M_3+i},$$

while for any other state  $\alpha' = (x', y')$  such that  $x' = x, y'_j \leq y_j$  for all  $j$  and  $\sum_j (y_j - y'_j) \geq 2$ ,

$$A(\alpha, \alpha') = \sum_{i=1}^{M_3} r_B^*(i) \varphi_B(i, y; y') x_{M_1+M_2+i}$$

$$+ \sum_{\ell=1}^N y_{\ell} \sum_{i=1}^{M_4} d_B^*(i, \ell) \mu_B(i, \ell, y; y') x_{M_1+M_2+M_3+i}.$$

Moreover, for all  $\alpha' \neq \alpha$  and not of the forms above, we have

$$A(\alpha, \alpha') = 0$$

and, finally,

$$A(\alpha, \alpha) = - \sum_{\alpha' \neq \alpha} A(\alpha, \alpha'). \quad \square$$

We omit the proof of the Theorem, which one constructs by using the results and by the methods of the appendix of KARR [8]. The expressions for the generator  $A$  have probabilistic interpretations and are written in what we feel is the most revealing form. Consider, for example, the term  $A(\alpha, \alpha')$  of Equation (3). Here the first summand (of four) on the right is the instantaneous rate at which Blue type- $i$  weapons are being killed by Red IS weapons when the two forces have compositions  $x$  and  $y$ , respectively, and the second term is a similar rate arising from Red PS weapons. For kills caused by single-shot weapons, "rate of kill" and "rate of kill one at a time" are the same. This is not so for multiple-shot weapons. Hence, the third summand on the right-hand side of Equation (3) must be interpreted as the instantaneous rate at which Blue type- $i$  weapons are being killed *exactly one at a time* by Red IM weapons. The interpretation of the fourth summand is then analogous; note that the

kill of a type-i weapon can arise, in principle, from the detection of any type of weapon. Hence  $A(\alpha, \alpha')$  is--when the structure of the two forces is  $\alpha = (x, y)$ --the instantaneous rate at which Blue type-i weapons are being destroyed precisely one at a time. We emphasize that single kills, in general, can arise from multiple-shot weapons, as indicated by the presence of the third and fourth summands in Equation (3).

For the term of  $A$  given by Equation (4), only multiple-kill weapons need be considered, and interpretations are similar to those given above.

The expressions  $A(\alpha, \alpha')$  given in the Theorem are valid only if  $\alpha$  is not an absorbing state. In this process the absorbing states are those of the forms  $(0, y)$  and  $(x, 0)$ , which correspond to annihilation of the Blue and Red sides, respectively. For such states  $\alpha$ ,  $A(\alpha, \alpha') = 0$  for all  $\alpha'$ .

We next list, for purposes of reference and completeness, some consequences of the Theorem.

**COROLLARIES.** (a) For the Markov process  $(B, R)$ , all states of the form  $(x, 0)$  with  $x \in \mathbb{N}^M$  or of the form  $(0, y)$  with  $y \in \mathbb{N}^N$  are absorbing; all other states are stable and transient;

(b) If  $\alpha \neq (0, 0)$ , then

$$P^\alpha\{(B_t, R_t) = (0, 0) \text{ for some } t\} = 0 ;$$

(c) For any  $\alpha$ , with  $P^\alpha$  - probability 1, each component of the sample paths  $t \rightarrow B_t$  and  $t \rightarrow R_t$  is nonincreasing. (There is no provision for reinforcements; one indication of how this problem might be handled is given in [8].)

For simplicity we have not written down explicit expressions for the jump rates  $\lambda(\alpha)$  or the transition matrix  $Q$  of the embedded Markov chain. These may be obtained in a straightforward manner from the infinitesimal generator using the formulae given before the Theorem.

The great generality of the Theorem possibly obscures its usefulness. In the next section we illustrate that usefulness by setting forth in some detail a number of special cases.

### 3. SOME SPECIFIC LANCHESTER PROCESSES

The objective of this section is to illustrate the scope and utility of the attrition process formulated in the previous section by deriving (as special cases) a number of simplified stochastic attrition processes. Some of these processes are known, principally in the context of stochastic analogues of the deterministic attrition models of F. W. LANCHESTER and many others; the reader is referred to [5] and [7] for surveys of this work. We point out such connections in this section not to justify the work reported here but to indicate its relation to previous research in the same general area. In particular, we believe that our assumptions and weapon classifications should be taken on their own merits and judged neither as stochastic analogues used to justify Lanchester's equations nor as a stochastic interpretation of the classical "linear law/square law" dichotomy. Applicability of our assumptions to any given physical situation is independent of such questions.

The reader should not infer from the discussion above either that there is no relation between stochastic and deterministic models of attrition or that the relation is uninteresting or unimportant. In fact, the relation is both interesting and important; details of it are explored in [12].

For each process below we indicate how to obtain it as a special case of the attrition process described in Section 2, present explicitly the infinitesimal generator  $A$ , jump function  $\lambda$  and transition matrix  $Q$  of the embedded Markov chain, and discuss the relation of the process to previously published

work. We begin with the simplest case of homogeneous forces, in which there is but one type of weapon on each side.

### Homogeneous Processes

Throughout this discussion we assume that there is only one type of weapon on each side. The attrition process is denoted by  $((B_t, R_t))_{t>0}$ , where  $B_t$  is the number of Blue weapons surviving at time  $t$  and  $R_t$  is the number of Red weapons surviving at time  $t$ , and has as its state space  $E = \mathbb{N} \times \mathbb{N}$ . Notational dependences of parameters of Section 2 on weapon type are suppressed for simplicity.

1. Independent Engagement Initiation, Single Kills. In this case times between engagements initiated by a Blue weapon are independent and identically exponentially distributed with expectation  $1/r_B$  and fatal with probability  $p_B$ ; corresponding parameters for Red are  $r_R$  and  $p_R$ , respectively. Then, the attrition process has infinitesimal generator  $A$  given by

$$\begin{aligned} A((i,j), (i,j-1)) &= ir_B p_B \\ A((i,j), (i,j)) &= - (ir_B p_B + jr_R p_R) \\ A((i,j), (i-1,j)) &= jr_R p_R , \end{aligned}$$

Jump function  $\lambda$  given by

$$\lambda(i,j) = ir_B p_B + jr_R p_R ,$$

and transition matrix  $Q$  of the embedded Markov chain given by

$$\begin{aligned} Q((i,j), (i,j-1)) &= \frac{ir_B p_B}{ir_B p_B + jr_R p_R} \\ Q((i,j), (i-1,j)) &= \frac{jR_R p_R}{ir_B p_B + jr_R p_R} . \end{aligned}$$

These expressions hold only for  $i > 0, j > 0$ . All other states are absorbing.

This particular attrition process was proposed by R. N. SNOW [13] as a stochastic analogue of Lanchester's square law of attrition.

2. Proportional Engagement Initiation, Single Kills. All weapons of the single type on each side are of class PM. The expected times to make a detection and kill probabilities are  $d_B^{-1}, k_B$  and  $d_R^{-1}, k_R$  for Blue and Red weapons, respectively. In this case the attrition process  $((B_t, R_t))_{t \geq 0}$  has infinitesimal generator A given by

$$\begin{aligned} A((i,j), (i,j-1)) &= ij d_B k_B \\ A((i,j), (i,j)) &= -ij(d_B k_B + d_R k_R) \\ A((i,j), (i-1,j)) &= ij d_R k_R, \end{aligned}$$

jump function  $\lambda$  given by

$$\lambda(i,j) = ij(d_B k_B + d_R k_R),$$

and transition matrix Q of the embedded Markov chain given by

$$\begin{aligned} Q((i,j), (i,j-1)) &= \frac{d_B k_B}{d_B k_B + d_R k_R} \\ Q((i,j), (i-1,j)) &= \frac{d_R k_R}{d_B k_B + d_R k_R}. \end{aligned}$$

The infinitesimal generator of this process is similar to the differential model known as Lanchester's linear law, so one could interpret within the Lanchesterean context this process as a stochastic linear law process. From a computational standpoint the embedded Markov chain of this process is particularly tractable: it is a spatially homogeneous random walk on the 2-dimensional lattice  $\mathbb{N} \times \mathbb{N}$ ; considerable theoretical and computational knowledge exists concerning such processes. For example, the probability that a given side wins according to criterion that involves only numbers of weapons and the

associated terminal distribution of the process can be computed using only the embedded Markov chain.

3. Asymmetric Engagement Initiation, Single Kills. The single Blue weapon type is of class IS with parameters  $r_B, p_B$ ; the single Red weapon type is of class PS with parameters  $d_R, k_R$ . For the resultant Markov attrition process the infinitesimal generator A is given by

$$\begin{aligned} A((i,j), (i,j-1)) &= ir_B p_B \\ A((i,j), (i,j)) &= - (ir_B p_B + ijd_R k_R) \\ A((i,j), (i-1,j)) &= ijd_R k_R , \end{aligned}$$

the jump function  $\lambda$  is given by

$$\lambda(i,j) = ir_B p_B + ijd_R k_R ,$$

and the transition matrix Q of the embedded Markov chain is given by

$$\begin{aligned} Q((i,j), (i,j-1)) &= \frac{r_B p_B}{r_B p_B + jd_R k_R} \\ Q((i,j), (i-1,j)) &= \frac{jd_R k_R}{r_B p_B + jd_R k_R} . \end{aligned}$$

Observe that  $Q((i,j), \cdot)$  is independent of  $i$ , which leads to simplifications of certain computations. In Lanchesterean terms, this process is a stochastic analogue of the mixed law process proposed by S. DEITCHMAN [4] as a model of guerilla ambushes.

4. Independent Engagement Initiation, Binomially Distributed Multiple Kills. Here the single type of weapon on each side belongs to class IM; the exponential firing rates are  $r_B^*$  and  $r_R^*$  for the Blue and Red sides, respectively. The kill distributions of Assumptions 10 and 11 are binomial. That is,

$$\varphi_B(j; k) = P\{k \text{ Red weapons survive an engagement initiated by a Blue weapon at the start of which } j \text{ Red weapons are surviving}\}$$

$$= \binom{j}{k} q_B^{j-k} (1-q_B)^k ,$$

where  $0 < q_B < 1$ . That is, each currently surviving Red weapon is killed with probability  $q_B$ , independently of all the others. We denote by  $q_R$  the analogously defined kill probability for weapons on the Red side. The Markov attrition process then has infinitesimal generator  $A$  given by

$$A((i, j), (i, \ell)) = ir_B^* \binom{j}{\ell} (1-q_B)^\ell q_B^{j-\ell} , \quad \ell = 0, \dots, j-1$$

$$A((i, j), (i, j)) = - [ir_B^*(1-(1-q_B)^j) + jr_R^*(1-(1-q_R)^i)]$$

$$A((i, j), (k, j)) = jr_R^* \binom{i}{k} (1-q_R)^k q_R^{i-k} , \quad k = 0, \dots, i-1 ,$$

jump function  $\lambda$  given by

$$\lambda(i, j) = ir_B^*(1-(1-q_B)^j) + jr_R^*(1-(1-q_R)^i) ,$$

and transition matrix  $Q$  of the embedded Markov chain given by

$$Q((i, j), (i, \ell)) = \frac{ir_B^* \binom{j}{\ell} (1-q_B)^\ell q_B^{j-\ell}}{\lambda(i, j)} , \quad \ell = 0, \dots, j-1$$

$$Q((i, j), (k, j)) = \frac{jr_R^* \binom{i}{k} (1-q_R)^k q_R^{i-k}}{\lambda(i, j)} , \quad k = 0, \dots, i-1 .$$

An attempt to classify this process as a "Lanchester square" or a "Lanchester linear" process helps to illustrate not only the difficulties but also the irrelevance of the classification. On the one hand the form of the jump function  $\lambda$  is similar to coefficients in the Lanchester square equations, but for small values of  $q_B$  and  $q_R$  the principal terms of the generator are those corresponding to one Red casualty: namely,

$$\begin{aligned}
 A((i,j), (i,j-1)) &= i r_B^j \binom{j}{j-1} (1-q_B)^{j-1} q_B \\
 &= i j r_B^j q_B (1-q_B)^{j-1}
 \end{aligned}$$

and to one Blue casualty: namely,

$$A((i,j), (i-1,j)) = i j r_R^j q_R (1-q_R)^{j-1} .$$

These latter terms clearly resemble the Lanchester linear equations, so no clear choice is possible. We again emphasize that within our formulation there is no need for such considerations; assumptions are always dealt with explicitly and on their own merits.

### Heterogeneous Processes

In the processes discussed below there are more than one type of weapon on at least one of the two sides. For such processes the parameters  $a_B, a_R$  (fire allocations for IS weapons),  $\varphi_B, \varphi_R$  (survivor distributions for IM weapons) and  $\mu_B, \mu_R$  (survivor distributions for PM weapons) must be specified; this is a problem on which our research has not concentrated. We have implicitly assumed that certain dependences can be neglected; for example, none of the above distributions depends on the current structure of the force to which the shooting weapon belongs. Below we indicate some possible specific forms for the distributions required. We hope others will contribute to this effort.

1. Independent Engagement Initiation, Uniform Fire Allocation, and Single Kills. All weapons on each side are of class IS, but each side may have several types of weapons. The engagement rates are  $r_B(i)$ ,  $r_R(j)$  for appropriate values of  $i$  and  $j$ ; the kill probabilities are  $p_B(i,j)$ ,  $p_R(j,i)$ . Fire allocation is uniform in the numerical sense and independent of the type of shooting weapon. That is, for all  $i, j$  and  $y$ ,

$$a_B(i, y; j) = y_j / \sum_k y_k ,$$

while for all  $i, j$  and  $x$

$$a_R(j, x; i) = x_i / \sum_k x_k .$$

The assumption of uniform fire allocation is plausible if different types of weapons are not significantly different in their characteristics as targets, but may differ in their characteristics as shooting weapons. The resultant attrition process has infinitesimal generator  $A$  given by

$$A((x, y); (x; y_1, \dots, y_{j-1}, \dots, y_{N_1})) = \frac{y_j}{\sum_k y_k} \sum_{i=1}^{M_1} d_B(i) p_B(i, j) x_i$$

$$A((x, y); (x, y)) = - \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} \left\{ \frac{y_j d_B(i) p_B(i, j) x_i}{\sum_k y_k} \right. \\ \left. + \frac{x_i d_R(j) p_R(j, i) y_j}{\sum_k x_k} \right\}$$

$$A((x, y); (x_1, \dots, x_{i-1}, \dots, x_{M_1}; y)) = \frac{x_i}{\sum_k x_k} \sum_{j=1}^{N_1} d_R(j) p_R(j, i) y_j ,$$

jump function  $\lambda$  given by

$$\lambda(x, y) = \sum_{i=1}^{M_1} \sum_{j=1}^{N_1} x_i y_j \left\{ \frac{d_B(i) p_B(i, j)}{\sum_k y_k} + \frac{d_R(j) p_R(j, i)}{\sum_k x_k} \right\} ,$$

and transition matrix  $Q$  of the embedded Markov chain given by

$$Q((x,y); (x; y_1, \dots, y_{j-1}, \dots, y_{N_1})) = \frac{y_j \sum_{i=1}^{M_1} d_B(i,j) p_B(i,j) x_i}{\lambda(x,y) \sum y_k}$$

$$Q((x,y); (x_1, \dots, x_{i-1}, \dots, x_{M_1}; y)) = \frac{x_1 \sum_{j=1}^{N_1} d_R(j,i) p_R(j,i) y_j}{\lambda(x,y) \sum x_k} .$$

We do not attempt a Lanchesterean classification of this process.

Some other methods of fire allocation are discussed in Section 4 below.

## 2. Proportional Engagement Initiation, Single Kills.

Weapons on each side are all of class PS, with expected times to detection denoted by  $d_B^{-1}(i,j)$ ,  $d_R^{-1}(j,i)$  and kill probabilities denoted by  $k_B(i,j)$ ,  $k_R(j,i)$ . The ranges of the indices are  $i = 1, \dots, M_2$  and  $j = 1, \dots, N_2$ . The attrition process that results is a straightforward extension of Process 2 in the subsection on homogeneous processes. The infinitesimal generator A is given by

$$A((x,y); (x; y_1, \dots, y_{j-1}, \dots, y_{N_2})) = y_j \sum_{i=1}^{M_2} d_B(i,j) k_B(i,j) x_i$$

$$A((x,y); (x,y)) = - \sum_{i=1}^{M_2} \sum_{j=1}^{N_2} x_i y_j [d_B(i,j) k_B(i,j) + d_R(j,i) k_R(j,i)]$$

$$A((x,y); (x_1, \dots, x_{i-1}, \dots, x_{M_2}; y)) = x_1 \sum_{j=1}^{N_2} d_R(j,i) k_R(j,i) y_j ,$$

and other parameters may be computed in the manner indicated in Section 2. For the sake of brevity we omit further details.

3. Independent Engagement Initiation, Uniform Fire Allocation, Binomially Distributed Kills of a Single Weapon Type. This example illustrates how, for weapons of class IM in the heterogeneous case, the kill distributions  $\varphi_B$ ,  $\varphi_R$  can be constructed so as to represent the effects of both allocation of fire and kill probabilities. We assume here that all weapons are of class IM, that fire allocation is uniform with respect to weapon type and that kills within a type of weapon are binomially distributed. That is, for each  $i$  and  $j$ ,

$$\varphi_B(i, y; (y_1, \dots, n, \dots, y_{N_3})) = \frac{y_j}{\sum y_\ell} \binom{y_j}{n} (1-q_B(i, j))^n q_B(i, j)^{y_j - n}$$

for  $n = 0, \dots, y_j$ . The interpretation is this: when a Blue weapon of type  $i$  initiates an engagement, the type  $j$  of the weapons to be engaged is chosen uniformly on the basis of the numbers of opposition weapons of all types present; the number of type- $j$  weapons destroyed is binomially distributed with parameters  $(y_j, q_B(i, j))$ . A physical situation that may be compatible would involve ammunitions that are highly target specific or target weapons physically grouped by type. Denote by  $q_R(j, 1)$  the corresponding individual weapon kill probabilities for Red. The resulting Markov attrition process is an amalgamation of Homogeneous Process 4 and Heterogeneous Process 1 and has infinitesimal generator  $A$  given by

$$A((x, y); (x; y_1, \dots, y_{j-1}, n, y_{j+1}, \dots, y_{N_3}))$$

$$= \frac{y_j}{\sum y_k} \sum_{i=1}^{M_3} r_B^*(i) \binom{y_j}{n} (1-q_B(i, j))^n q_B(i, j)^{y_j-n} x_i$$

$$A((x, y); (x, y)) = - \sum_{i=1}^{M_3} \sum_{j=1}^{N_3} x_i y_j \left[ \frac{d_B^*(i) [1 - (1-q_B(i, j))^{y_j}]}{\sum y_k} \right. \\ \left. + \frac{d_R^*(j) [1 - (1-q_R(j, i))^{x_i}]}{\sum x_k} \right]$$

$$A((x, y); (x_1, \dots, x_{i-1}, m, x_{i+1}, \dots, x_{M_3}; y))$$

$$= \frac{x_i}{\sum x_k} \sum_{j=1}^{M_3} r_R^*(j) \binom{x_i}{m} (1-q_R(j, i))^m q_R(j, i)^{x_i-m} y_j .$$

These expressions are valid for  $0 \leq n < y_j$  and  $0 \leq m < x_i$ . Again for the sake of brevity, we omit the corresponding expressions for the jump function and the transition matrix of the embedded Markov chain.

The family of processes that can be constructed in the manner we have used above is limitless, of course; the examples above serve only to illustrate what can be done in the simplest cases. In Section 4 we consider the representation of some other specific effects of interest.

#### 4. FURTHER ASPECTS

This section contains additional discussion on a number of points raised in preceding sections.

##### Computational Problems

In order to apply the models derived here to computerized combat simulations one must develop means of approximating expected numbers of survivors at various times. To indicate how this problem is approached within the context of the model of Section 2 without obscuring the discussion by unnecessary generality, let us restrict attention to a homogeneous process. Let  $P^{(i,j)}$  denote the probability law of the process with initial conditions  $B_0 = i$ ,  $R_0 = j$ ,  $E^{(i,j)}$  the corresponding expectation operator, and  $(P_t)$  the transition function of the process. The main problem is computation or approximation of the expected number of Blue weapons surviving at a fixed time  $t$ , namely

$$E^{(i,j)}[B_t] = \sum_{k=1}^i k \sum_{l=1}^j P_t((i,j),(k,l)) ;$$

the approximation of  $E^{(i,j)}[R_t]$  is entirely analogous. The given data are the entries of the infinitesimal generator  $A$ . It follows from the backward and forward equations that for each  $t$ ,

$$P_t = e^{tA} = \sum_{n=0}^{\infty} t^n A^n / n! . \quad (6)$$

This computation is usually impossible to carry out in closed form, but one may approximate the infinite sum by various

finite sums. For example, for each  $(i, j)$  and  $(k, \ell)$

$$P_t((i, j), (k, \ell)) \sim I((i, j), (k, \ell)) + tA((i, j), (k, \ell))$$

so that, consequently,

$$E^{(i, j)}[B_t] \sim i + t \sum_{k=0}^1 \sum_{\ell=0}^j A((i, j), (k, \ell)) . \quad (7)$$

To illustrate the meaning of (7), we list in Table 3 below the specific form it takes for each of the four homogeneous processes presented in Section 3.

Table 3. APPROXIMATIONS TO  $E[B_t]$

Process	Expression (7) for This Process
1	$E^{(i, j)}[B_t] \sim i - jr_R p_R t$
2	$E^{(i, j)}[B_t] \sim i - ijd_R k_R t$
3	$E^{(i, j)}[B_t] \sim i - ijd_R k_R t$
4	$E^{(i, j)}[B_t] \sim i - ijr_R^* q_R t$

The approximations generated by (7) are computationally useful in that no storage requirements and no long calculations are involved. The same need not be true, for example, of the second order approximation

$$E^{(i, j)}[B_t] \sim i + t \sum_{k=0}^1 \sum_{\ell=0}^j A((i, j), (k, \ell)) + \frac{t^2}{2} \sum_{k=0}^1 \sum_{\ell=0}^j A^2((i, j), (k, \ell)) . \quad (8)$$

So long as the generator matrix  $A$  is available in closed form (as it is in all the processes obtainable within the general setting of Section 2) the storage requirements for  $A$  itself are minimal. But  $A^2$  may not admit a simple closed form expression and so may have to be stored in a computer in order to make use of (8); the burden could be significant. In addition, substantial computation time is needed to calculate  $A^2$  if the number of weapon types is even moderately large. Finally, other assumptions may negate the better degree of approximation of (8) as opposed to (7). For purposes of comparison we give in Table 4 the approximation to  $E^{(i,j)}[B_t]$  obtained by applying (8) to the first two homogeneous processes described in Section 3.

Table 4. FURTHER APPROXIMATIONS TO  $E[B_t]$

Process	Expression (8) for This Process
1	$E^{(i,j)}[B_t] \sim i - jr_R p_R t + ir_B p_B r_R p_R \frac{t^2}{2}$
2	$E^{(i,j)}[B_t] \sim i - ijd_R k_R t + i^2 j d_B k_B d_R k_R \frac{t^2}{2}$ + $ij^2 d_R^2 k_R^2 \frac{t^2}{2}$

A heuristic justification of the first entry in Table 4 is the following: the first order estimate

$$E^{(i,j)}[B_t] \sim i - jr_R p_R t \quad (9)$$

in fact overestimates Blue losses, since Red strength does not remain at  $j$  throughout the time interval  $[0,t]$ . Using the analogous estimate

$$E^{(i,j)}[R_t] \sim j - ir_B p_B t,$$

one can take as a measure of Red strength the average

$$\hat{j} = \frac{1}{2}[j + (j - ir_B p_B t)] = j - \frac{ir_B p_B t}{2}; \quad (10)$$

replacement of  $j$  by  $\hat{j}$  in (9) yields the first expression in Table 4. The expression for Process 2 is derived in an analogous manner.

The point is that from a computational standpoint the first order approximation (7) is always feasible and is directly available from the infinitesimal generator of the process. Higher order approximations may or may not be computationally feasible; if such approximations can be obtained in closed form (as in Table 4) their use is probably justified. Within theater-level modeling contexts use of approximations of third degree or higher is almost certainly as unnecessary as it is difficult.

Another still largely unexplored computational tool is the embedded Markov chain associated with the attrition process under study. Distributions, expectations and variances of random variables defined in terms of the attrition process, but not explicitly involving the continuous time parameter, can be calculated using well-developed computational methods for Markov chains. For example, suppose there is defined a fixed subset  $A$  of the state space  $E$  which is a termination set in the sense that the engagement terminates at the random time  $T$  at which the process  $((B_t, R_t))_{t \geq 0}$  first enters  $A$ . Then the distribution, expectation, and variance of the terminal state  $(B_T, R_T)$  could be obtained using Markov chain methods. The length  $T$  of the engagement could not be so treated. For some examples of such analyses the reader is referred to [10] and [11].

### Fire Allocation

As mentioned in Section 3, derivation of the fire allocation distributions  $a_B, a_R$  for weapons of class IS is a problem of considerable practical interest; in the heterogeneous processes discussed there we considered only the uniform fire allocation

$$a_B(i, y; j) = y_j / \sum_l y_l .$$

This particular form of allocation is not always appropriate so we shall briefly discuss some alternatives; these illustrate the variety of effects that can be represented within the context of our attrition process. For the rest of the discussion we confine ourselves to a particular type  $i$  of Blue weapons.

1. Restricted Uniform Allocation. Suppose there exists a subset  $K_1$  of the index set  $\{1, \dots, N\}$  of Red weapon types such that a Blue weapon of type  $i$  will never engage a Red weapon of type  $j$  if  $j \notin K_1$ ; such Red weapons may be interpreted as invulnerable to Blue weapons of type  $i$ . Otherwise, the fire allocation is uniform. Then we would have

$$a_B(i, y; j) = \begin{cases} y_j / \sum_{l \in K_1} y_l & \text{if } j \in K_1 \\ 0 & \text{otherwise} . \end{cases}$$

2. Priority Fire Allocation. Suppose that the  $N$  Red weapon types are (unambiguously) ranked in some order  $(j_1(i), \dots, j_N(i))$ , where  $(j_1(i), \dots, j_N(i))$  is simply a permutation of  $(1, \dots, N)$ . Further suppose that a Red weapon of type  $j_k(i)$  will never be engaged if a Red weapon of type  $j_{k-1}(i)$  is present. This would be represented by

$$a_B(i, y; j_{k-1}(i)) = 1 ,$$

where

$$k^* = \min\{k: y_{j_k(i)} \neq 0\}.$$

The type- $i$  weapons concentrate their fire upon the highest priority opposition weapons currently surviving.

3. An Axiomatic Fire Allocation. We note here that all allocations satisfying a simple set of axioms are of a particular form. The axioms are:

- (i)  $a_B(i, y; j) = 0$  if and only if  $y_j = 0$ ;
- (ii) For all target forces  $y$  and  $z$  and all  $j$ ,

$$a_B(i, y+z; j) = a_B(i, y; j) \left[ \sum_{\ell=1}^N \frac{y_\ell}{y_\ell + z_\ell} a_B(i, y+z; \ell) \right] + a_B(i, z; j) \left[ \sum_{\ell=1}^N \frac{z_\ell}{y_\ell + z_\ell} a_B(i, y+z; \ell) \right].$$

We interpret these axioms in physical terms as follows. The first states that a weapon type not present receives no fire but that every weapon type present receives a positive fraction of the fire. The second is best explained step by step. Consider the effect of combining two target forces  $y$  and  $z$  into the single force  $w = y + z$ .  $a_B(i, y+z; \ell)$  is the proportion of fire directed at the combined force  $w$  which is allocated to type  $\ell$  weapons. If this fire is further allocated among the type  $\ell$  weapons from the two component forces  $y$  and  $z$  in proportion to the relative numbers of such weapons present, then

$$\frac{y_\ell}{y_\ell + z_\ell} a_B(i, y+z; \ell)$$

is the fraction of fire directed at the combined force that is allocated to type  $\ell$  weapons originally part of the  $y$ -force.

Thus

$$\alpha_y = \sum_{\ell=1}^N \frac{y_\ell}{y_\ell + z_\ell} a_B(i, y+z; \ell)$$

is the proportion of fire allocated to weapons of types originally in the  $y$ -force and

$$\alpha_z = \sum_{\ell=1}^N \frac{z_\ell}{y_\ell + z_\ell} a_B(i, y+z; \ell)$$

the fire allocated to weapons originally in the  $z$ -force. But the fire represented by  $\alpha_y$  should be allocated among weapon types according to the distribution  $a_B(i, y; \cdot)$  and similarly for  $\alpha_z$  so that one should have

$$a_B(i, y+z; \cdot) = \alpha_y a_B(i, y; \cdot) + \alpha_z a_B(i, z; \cdot) .$$

If an assumption of this form did not hold, consistency problems would arise, with fire allocation dependent on names given targets rather than only numbers of targets. (For example, arbitrarily splitting a class of  $n$  indistinguishable weapons into two subclasses of  $n_1$  "Type A" and  $n - n_1$  "Type B" weapons would change the fire allocation.)

It has been shown by J. BLANKENSHIP [1] that the axioms (i) and (ii) imply that

$$a_B(i, y; j) = b_j y_j / \sum_{\ell=1}^N b_\ell y_\ell ,$$

where  $b_j = a_B(i, (1, \dots, 1); j)$ . In this case, the fire allocation for any target force is specified by that for any given base force, and is linear in numbers of target weapons.

### Kill Distributions

Analysis of the sort used to define different forms of fire allocation is also necessary to derive plausible specific forms for the kill distributions for weapons of classes IM and PM, especially the former. Unfortunately, little such analysis has been carried out. In particular, some dependences allowed by the notations can probably be neglected. For example, weapons of class PM might be allowed to kill only weapons of the same type as the weapon whose detection initiated the engagement. If the binomial distributions of Homogeneous Process 4 were felt to be applicable then we would have

$$\mu_B(i, j, y; z) = \binom{y_j}{\ell} (1-q_B^*(i, j))^{\ell} q_B^*(i, j)^{y_j - \ell}$$

for  $z = (y_1, \dots, y_{j-1}, \ell, y_{j+1}, \dots, y_N)$  and  $\ell = 0, \dots, y_j$ . Further work along these lines is clearly necessary in order to fully exploit the capabilities of our model.

### Further Phenomena

It is assumed throughout that all engagements occur instantaneously, with ensuing total loss of contact. A method for relaxing this assumption, but retaining the Markov property of the attrition process, is discussed in [8]. In [8] a discussion is also given of possible inclusion of randomly arriving reinforcements. Neither of these problems, however, is of the same interest or importance as derivation of kill distributions for multiple-kill weapons.

A final problem, which could turn out to be of considerable significance, concerns dependences not allowed in the process of Section 2. None of the parameters depends on the structure of the force to which the weapon initiating an engagement belongs. A plausible representation of such dependence would permit modeling of synergistic relations among weapons on the same side, a phenomenon believed to be of considerable importance but so far not well treated in attrition models.

#### ACKNOWLEDGMENTS

This paper is based on [8] and [9], which contain proofs of the results presented here. The research leading to [8] and [9] was funded by the Independent Research Program of the Institute for Defense Analyses. The author is grateful to Drs. L. B. ANDERSON, J. BRACKEN, J. McGILL and J. TAYLOR for numerous helpful criticisms and comments. Currently the author is Assistant Professor of Mathematical Sciences at The Johns Hopkins University.

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